1.4 Problems

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

1. $\frac{dy}{dx} + 2xy = 0$

2. $\frac{dy}{dx} + 2xy^2 = 0$

3. $\frac{dy}{dx} = y \sin x$

4. $(1+x)\frac{dy}{dx} = 4y$

5. $2\sqrt{x}\,\frac{dy}{dx} = \sqrt{1-y^2}$

6. $\frac{dy}{dx} = 3\sqrt{xy}$

7. $\frac{dy}{dx} = (64xy)^{1/3}$

8. $\frac{dy}{dx} = 2x \sec y$

9. $(1-x^2)\frac{dy}{dx} = 2y$

10. $(1+x)^2\frac{dy}{dx} = (1+y)^2$

11. $y' = xy^3$

12. $yy' = x(y^2+1)$

13. $y^3\frac{dy}{dx} = (y^4+1)\cos x$

14. $\frac{dy}{dx} = \frac{1+\sqrt{x}}{1+\sqrt{y}}$

15. $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}$

16. $(x^2+1)(\tan y)y' = x$

17. $y' = 1+x+y+xy$ (*Suggestion*: Factor the right-hand side.)

18. $x^2y' = 1 - x^2 + y^2 - x^2y^2$

Find explicit particular solutions of the initial value problems in Problems 19 through 28.

19. $\frac{dy}{dx} = ye^x, \quad y(0) = 2e$

20. $\frac{dy}{dx} = 3x^2(y^2+1), \quad y(0) = 1$

21. $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}, \quad y(5) = 2$

22. $\frac{dy}{dx} = 4x^3y - y, \quad y(1) = -3$

23. $\frac{dy}{dx} + 1 = 2y, \quad y(1) = 1$

24. $(\tan x)\frac{dy}{dx} = y, \quad y\left(\tfrac{1}{2}\pi\right) = \tfrac{1}{2}\pi$

25. $x\frac{dy}{dx} - y = 2x^2y, \quad y(1) = 1$

26. $\frac{dy}{dx} = 2xy^2 + 3x^2y^2, \quad y(1) = -1$

27. $\frac{dy}{dx} = 6e^{2x-y}, \quad y(0) = 0$

28. $2\sqrt{x}\,\frac{dy}{dx} = \cos^2 y, \quad y(4) = \pi/4$

29. **(a)** Find a general solution of the differential equation $dy/dx = y^2$. **(b)** Find a singular solution that is not included in the general solution. **(c)** Inspect a sketch of typical solution curves to determine the points (a, b) for which the initial value problem $y' = y^2$, $y(a) = b$ has a unique solution.

30. Solve the differential equation $(dy/dx)^2 = 4y$ to verify the general solution curves and singular solution curve that are illustrated in Fig. 1.4.5. Then determine the points (a, b) in the plane for which the initial value problem $(y')^2 = 4y$, $y(a) = b$ has **(a)** no solution, **(b)** infinitely many solutions that are defined for all x, **(c)** on some neighborhood of the point $x = a$, only finitely many solutions.

31. Discuss the difference between the differential equations $(dy/dx)^2 = 4y$ and $dy/dx = 2\sqrt{y}$. Do they have the same solution curves? Why or why not? Determine the points (a, b) in the plane for which the initial value problem $y' = 2\sqrt{y}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

32. Find a general solution and any singular solutions of the differential equation $dymyslashdx = y\sqrt{y^2-1}$. Determine the points (a, b) in the plane for which the initial value problem $y' = y\sqrt{y^2-1}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

33. (Population growth) A certain city had a population of 25000 in 1960 and a population of 30000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

34. (Population growth) In a certain culture of bacteria, the number of bacteria increased sixfold in 10 h. How long did it take for the population to double?

35. (Radiocarbon dating) Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

36. (Radiocarbon dating) Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present-day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity?

37. (Continuously compounded interest) Upon the birth of their first child, a couple deposited \$5000 in an account that pays 8% interest compounded continuously. The interest payments are allowed to accumulate. How much will the account contain on the child's eighteenth birthday?

38. (Continuously compounded interest) Suppose that you discover in your attic an overdue library book on which your grandfather owed a fine of 30 cents 100 years ago. If an overdue fine grows exponentially at a 5% annual rate compounded continuously, how much would you have to pay if you returned the book today?

39. (Drug elimination) Suppose that sodium pentobarbital is used to anesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 milligrams (mg) of sodium pentobarbitol per kilogram of the dog's body

weight. Suppose also that sodium pentobarbitol is eliminated exponentially from the dog's bloodstream, with a half-life of 5 h. What single dose should be administered in order to anesthetize a 50-kg dog for 1 h?

40. The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the level acceptable for human habitation. How long will it be until the region is again habitable? (Ignore the probable presence of other radioactive isotopes.)

41. Suppose that a mineral body formed in an ancient cataclysm—perhaps the formation of the earth itself—originally contained the uranium isotope ^{238}U (which has a half-life of 4.51×10^9 years) but no lead, the end product of the radioactive decay of ^{238}U. If today the ratio of ^{238}U atoms to lead atoms in the mineral body is 0.9, when did the cataclysm occur?

42. A certain moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half-life is about 1.28×10^9 years) and that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium?

43. A pitcher of buttermilk initially at 25°C is to be cooled by setting it on the front porch, where the temperature is 0°C. Suppose that the temperature of the buttermilk has dropped to 15°C after 20 min. When will it be at 5°C?

44. When sugar is dissolved in water, the amount A that remains undissolved after t minutes satisfies the differential equation $dA/dt = -kA$ $(k > 0)$. If 25% of the sugar dissolves after 1 min, how long does it take for half of the sugar to dissolve?

45. The intensity I of light at a depth of x meters below the surface of a lake satisfies the differential equation $dI/dx = (-1.4)I$. **(a)** At what depth is the intensity half the intensity I_0 at the surface (where $x = 0$)? **(b)** What is the intensity at a depth of 10 m (as a fraction of I_0)? **(c)** At what depth will the intensity be 1% of that at the surface?

46. The barometric pressure p (in inches of mercury) at an altitude x miles above sea level satisfies the initial value problem $dp/dx = (-0.2)p$, $p(0) = 29.92$. **(a)** Calculate the barometric pressure at 10,000 ft and again at 30,000 ft. **(b)** Without prior conditioning, few people can survive when the pressure drops to less than 15 in. of mercury. How high is that?

47. A certain piece of dubious information about phenylethylamine in the drinking water began to spread one day in a city with a population of 100,000. Within a week, 10,000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population of the city has heard the rumor?

48. According to one cosmological theory, there were equal amounts of the two uranium isotopes ^{235}U and ^{238}U at the creation of the universe in the "big bang." At present there are 137.7 atoms of ^{238}U for each atom of ^{235}U. Using the half-lives 4.51×10^9 years for ^{238}U and 7.10×10^8 years for ^{235}U, calculate the age of the universe.

49. A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

50. The amount $A(t)$ of atmospheric pollutants in a certain mountain valley grows naturally and is tripling every 7.5 years.

(a) If the initial amount is 10 pu (pollutant units), write a formula for $A(t)$ giving the amount (in pu) present after t years.

(b) What will be the amount (in pu) of pollutants present in the valley atmosphere after 5 years?

(c) If it will be dangerous to stay in the valley when the amount of pollutants reaches 100 pu, how long will this take?

51. An accident at a nuclear power plant has left the surrounding area polluted with radioactive material that decays naturally. The initial amount of radioactive material present is 15 su (safe units), and 5 months later it is still 10 su.

(a) Write a formula giving the amount $A(t)$ of radioactive material (in su) remaining after t months.

(b) What amount of radioactive material will remain after 8 months?

(c) How long—total number of months or fraction thereof—will it be until $A = 1$ su, so it is safe for people to return to the area?

52. There are now about 3300 different human "language families" in the whole world. Assume that all these are derived from a single original language, and that a language family develops into 1.5 language families every 6 thousand years. About how long ago was the single original human language spoken?

53. Thousands of years ago ancestors of the Native Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, they have fanned out across North and South America. The single language that the original Native Americans spoke has since split into many Indian "language families." Assume (as in Problem 52) that the number of these language families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's Native Americans arrive?

54. A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft, and a bottom plug is removed at time $t = 0$ (hours). After 1 h the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank?

55. Suppose that the tank of Problem 48 has a radius of 3 ft and that its bottom hole is circular with radius 1 in. How

1.5 Problems

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

1. $y' + y = 2$, $y(0) = 0$
2. $y' - 2y = 3e^{2x}$, $y(0) = 0$
3. $y' + 3y = 2xe^{-3x}$
4. $y' - 2xy = e^{x^2}$
5. $xy' + 2y = 3x$, $y(1) = 5$
6. $xy' + 5y = 7x^2$, $y(2) = 5$
7. $2xy' + y = 10\sqrt{x}$
8. $3xy' + y = 12x$
9. $xy' - y = x$, $y(1) = 7$
10. $2xy' - 3y = 9x^3$
11. $xy' + y = 3xy$, $y(1) = 0$
12. $xy' + 3y = 2x^5$, $y(2) = 1$
13. $y' + y = e^x$, $y(0) = 1$
14. $xy' - 3y = x^3$, $y(1) = 10$
15. $y' + 2xy = x$, $y(0) = -2$
16. $y' = (1 - y)\cos x$, $y(\pi) = 2$
17. $(1 + x)y' + y = \cos x$, $y(0) = 1$
18. $xy' = 2y + x^3 \cos x$
19. $y' + y \cot x = \cos x$
20. $y' = 1 + x + y + xy$, $y(0) = 0$
21. $xy' = 3y + x^4 \cos x$, $y(2\pi) = 0$
22. $y' = 2xy + 3x^2 \exp(x^2)$, $y(0) = 5$
23. $xy' + (2x - 3)y = 4x^4$
24. $(x^2 + 4)y' + 3xy = x$, $y(0) = 1$
25. $(x^2 + 1)\dfrac{dy}{dx} + 3x^3y = 6x \exp\left(-\frac{3}{2}x^2\right)$, $y(0) = 1$

Solve the differential equations in Problems 26 through 28 by regarding y as the independent variable rather than x.

26. $(1 - 4xy^2)\dfrac{dy}{dx} = y^3$
27. $(x + ye^y)\dfrac{dy}{dx} = 1$
28. $(1 + 2xy)\dfrac{dy}{dx} = 1 + y^2$
29. Express the general solution of $dy/dx = 1 + 2xy$ in terms of the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

30. Express the solution of the initial value problem

$$2x\frac{dy}{dx} = y + 2x\cos x, \quad y(1) = 0$$

as an integral as in Example 3 of this section.

Problems 31 and 32 illustrate—for the special case of first-order linear equations—techniques that will be important when we study higher-order linear equations in Chapter 3.

31. **(a)** Show that

$$y_c(x) = Ce^{-\int P(x)\,dx}$$

is a general solution of $dy/dx + P(x)y = 0$. **(b)** Show that

$$y_p(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)e^{\int P(x)\,dx}\right)dx\right]$$

is a particular solution of $dy/dx + P(x)y = Q(x)$. **(c)** Suppose that $y_c(x)$ is any general solution of $dy/dx + P(x)y = 0$ and that $y_p(x)$ is any particular solution of $dy/dx + P(x)y = Q(x)$. Show that $y(x) = y_c(x) + y_p(x)$ is a general solution of $dy/dx + P(x)y = Q(x)$.

32. **(a)** Find constants A and B such that $y_p(x) = A\sin x + B\cos x$ is a solution of $dy/dx + y = 2\sin x$. **(b)** Use the result of part (a) and the method of Problem 31 to find the general solution of $dy/dx + y = 2\sin x$. **(c)** Solve the initial value problem $dy/dx + y = 2\sin x$, $y(0) = 1$.

33. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture—kept uniform by stirring— is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

34. Consider a reservoir with a volume of 8 billion cubic feet (ft^3) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft^3 of water with a pollutant concentration of 0.05% and an equal daily outflow of the well-mixed water in the reservoir. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?

35. Rework Example 4 for the case of Lake Ontario, which empties into the St. Lawrence River and receives inflow from Lake Erie (via the Niagara River). The only differences are that this lake has a volume of 1640 km^3 and an inflow-outflow rate of 410 km^3/year.

36. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 h. **(a)** Find the amount of salt in the tank after t minutes. **(b)** What is the maximum amount of salt ever in the tank?

37. A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

38. Consider the *cascade* of two tanks shown in Fig. 1.5.5, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1. **(a)** Find the amount $x(t)$ of salt in tank 1 at time t. **(b)** Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200},$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a). **(c)** Finally, find the maximum amount of salt ever in tank 2.